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# What is... Chern-Simons theory?

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This is an informal note on Chern-Simons theory (a.k.a. Jones-Witten theory) prepared for a talk at Caltech Graduate Student Seminar. The emphasis is on general overview and intuition, and there will be no proof at all. Interested readers may want to consult the listed references.

## 1 Introduction : What is a TQFT?

### 1.1 Axiomatic definition

Axiomatically [1], an $(n+1)$-*dim'l TQFT* $Z$ is simply a symmetric monoidal functor from $(n+1)Cob$ to $Vect$. [1]

$$Z : (n+1)Cob \longrightarrow Vect$$

More precisely [2], for each compact oriented smooth $n$-manifold $\Sigma$, $Z(\Sigma) = \mathcal{H}_\Sigma$ is a finite dimensional complex vector space, and for each compact oriented $(n+1)$-manifold $Y$ with boundary $\Sigma$, $Z(Y) \in Z(\Sigma)$. They satisfy the following axioms :

A1 (Involutory) $Z(\Sigma^*) = Z(\Sigma)^*$

A2 (Multiplicativity) $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

A3 (Associativity) If $Y_1$ is a cobordism from $\Sigma_1$ to $\Sigma_2$ and $Y_2$ is a cobordism from $\Sigma_2$ to $\Sigma_3$, then for the composite cobordism $Y = Y_1 \cup_{\Sigma_2} Y_2$, $Z(Y) = Z(Y_2)Z(Y_1) \in \mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_3))$.

as well as non-triviality axioms

A4 $Z(\emptyset) = \mathbb{C}$

A5 $Z(\Sigma \times I)$ is the identity endomorphism of $Z(\Sigma)$.

Some obvious consequences of these axioms are :

- If you cut a closed $(n+1)$-manifold into two pieces, say $Y_1$ and $Y_2$, along their common boundary $\Sigma$, then

  $$Z(Y) = \langle Z(Y_1), Z(Y_2) \rangle$$

  Basically this property makes TQFTs highly computable.
- Each $Z(\Sigma)$ carries a representation of $\mathrm{MCG}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\text{isotopy}$. [2]
- If $f \in \mathrm{MCG}(\Sigma)$, then

  $$Z(\Sigma_f) = \mathrm{Tr} Z(f)$$

  In particular, $Z(\Sigma \times S^1) = \dim Z(\Sigma)$.

[1] There are all sorts of variants, e.g. one can replace $Vect$ by any monoidal tensor category $C$, or decorate the TQFT with more structures on the manifolds. Also there are extended TQFTs which deal with higher categories [11].

[2] In actual physical TQFTs, we may only have projective representations.

### 1.2 TQFT as a 2-step "functor"

Physically, $Z(\Sigma) = \mathcal{H}_\Sigma$ are Hilbert spaces of states, and $Z(Y)$ are partition functions, determining (time-) evolution of states. TQFTs are the simplest kinds of QFTs, because there is no dynamics; the Hamiltonian is identically 0. There are lots of concrete examples of TQFT coming from physics, and they are roughly constructed through the following 2-step process [9] :

$$\boxed{\text{geometric structure}} \xrightarrow{\text{physics}} \boxed{\text{classical phase space}} \xrightarrow{\text{quantization}} \boxed{\text{quantum Hilbert space}}$$

1. Starting from a geometric structure, put a physical theory on that geometric structure, and solve the classical EOM. [3] The moduli space of boundary conditions is the classical phase space, which is a symplectic manifold.

2. Quantize the classical phase space (using geometric quantization [5]) to obtain the quantum Hilbert space.

As an example, suppose $M$ is a 3-manifold with boundary $\Sigma$. Let $G$ be a semisimple Lie group (e.g. $SU(2)$) and consider a physical theory on $M$ where fields $A$ are (gauge equivalent classes of) $G$-connections on $M$ and the Chern-Simons Lagrangian [4] is given by

$$CS(A) := \frac{k}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \in \mathbb{R}/\mathbb{Z}$$

The EOM is then $F_A = dA + A \wedge A = 0$; i.e. classical fields are flat connections. Hence the classical phase space is the moduli space of flat $G$-connections on the boundary $\Sigma$

$$\mathcal{M}_\Sigma$$

and this space (and $\mathcal{A}_\Sigma$) is equipped with a natural symplectic form

$$\omega(\alpha, \beta) = -\frac{1}{8\pi^2} \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta)$$

There is a natural restriction map from the moduli space of flat connections on the whole space to the classical phase space

$$\mathcal{M}_M \to \mathcal{M}_\Sigma$$

After geometric quantization, we get a Hilbert space of holomorphic sections of $\mathcal{L}^{\otimes k}$, where $\mathcal{L}$ is a complex line bundle over $\mathcal{M}_{\mathrm{flat}}(\Sigma, G)$, and a vector (= holomorphic section) determined by $M$. [8]

### 1.3 Schwarz type and Witten type

Physicists divide TQFTs into roughly two types [10] : Schwarz type TQFTs and Witten type TQFTs [6].

- *Schwarz type TQFTs* are those whose topological invariance is manifest directly from the Lagrangian. Examples include Chern-Simons theory [7], 2d Yang-Mills theory and BF theory

- *Witten type TQFTs* are those whose topological invariance is not manifest directly from the Lagrangian. Often, Witten type TQFTs have some supersymmetry (or a BRST operator) $Q$ thanks to which topological invariance is maintained. The Hilbert space in a Witten type TQFT is just the $Q$-cohomology. Examples of Witten type TQFTs include Seiberg-Witten theory and Floer homology.

In this talk, we focus on Chern-Simons theory, which is a Schwarz type theory.

[3] Often, a physical theory is specified by a description of "fields" and a choice of Lagrangian.

[4] This Chern-Simons functional may look arbitrary at first, but actually it arises naturally as the antiderivative of the 1-form $F$ on $\mathcal{A}/\mathcal{G}$. That is, $dCS(A) = \frac{1}{4\pi^2} F_A$.

## 2 WRT invariants

### 2.1 Definition

The *Witten-Reshetikhin-Turaev invariant* is an invariant of 3-manifolds, possibly with a (colored, oriented, framed) link inside it. There are many different ways to define the WRT invariant : using path integral [7], using conformal field theory and affine Lie algebras [3], or using quantum groups [4]. In this note, let's look at the path integral definition.

Let's fix a compact semisimple Lie group $G$ (gauge group) and a positive integer $k$ (level). Then for each closed 3-manifold $M$ with a link $L \subset M$ whose components $L_1, \cdots, L_r$ are colored by representations $R_1, \cdots, R_r$ of $G$, the WRT invariant of $(M, L)$ is a number

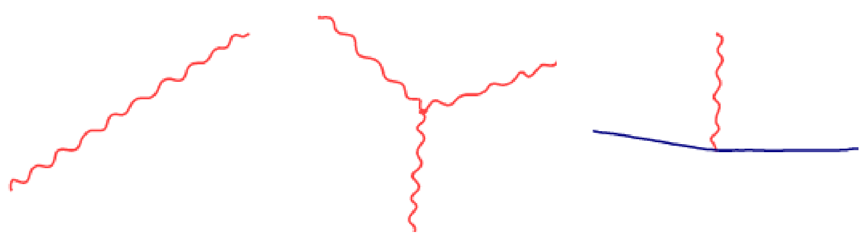
$$Z(M,L) := \int_{A \in \mathcal{A}/\mathcal{G}} \prod_{i=1}^{r} \mathrm{Tr}(\mathrm{Hol}_{L_i} R_i)\, e^{2\pi i k CS(A)} \mathcal{D}A \in \mathbb{C}$$

normalized in a way that $Z(S^2 \times S^1) = 1$. In particular, $J_L := \frac{Z(S^3,L)}{Z(S^3)}$ is a link invariant.

This a Feynman path integral. The moduli space $\mathcal{A}/\mathcal{G}$ of all fields is infinite dimensional, and no mathematically rigorous definition of path integral in general exists yet. But still we can get a lot of intuition from this path integral expression. For instance, in case of abelian CS, the path integral is just a Gaussian integral, and one can directly compute it. Also one can proceed perturbatively, and the perturbative CS is related to torsions of 3-manifolds and Vassiliev invariants of knots.

### 2.2 Heuristic meaning (Feynman diagrams)

Before we get to the actually meaning of this invariant, let's see what it means heuristically. In any QFT specified by a Lagrangian, you can get a lot of information by just directly reading off free part and interaction part. In case of Chern-Simons theory, the free part is $A \wedge dA$, and the interaction part is $\frac{2}{3} A \wedge A \wedge A$. Hence in Feynman diagrams, the free part will be corresponding to propagation of a gauge field, and the interaction part will be corresponding to trivalent vertices. Moreover, the Wilson lines play the role of source.



The path integral can be heuristically understood as the amplitude obtained by summing and integrating over all the possible Feynman diagrams.

### 2.3 How to compute it? (Jones polynomial and its generalization)

If the path integral is not rigorously defined, then how can we actually compute it? As we've seen in section 1, we can use the cutting and gluing! So once we have a description of a 3-manifold as surgery or Heegaard splitting, information of mapping class group representation is enough to compute the WRT invariant. In particular, when we have a surgery presentation of a manifold, we only need to know what is the representation of $SL(2, \mathbb{Z})$ given by the Chern-Simons theory. Let $\Sigma_1$ be a torus. A basis of $\mathcal{H}_{\Sigma_1}$ is given by a single Wilson line lying at the core of a solid torus, colored by certain representations of the quantum group $U_q(\mathfrak{g})$ [4] or the affine Lie algebra $\hat{\mathfrak{g}}$ [3]

capped by the level $k$; i.e. there are only finitely many colors, and the number of colors is exactly $\dim \mathcal{H}_{\Sigma_1}$. Then the problem reduces to finding values of $J_L = \frac{Z(S^3,L)}{Z(S^3)}$ for $L$ = an unknot with framing 1 and for $L$ = a Hopf link. [5]

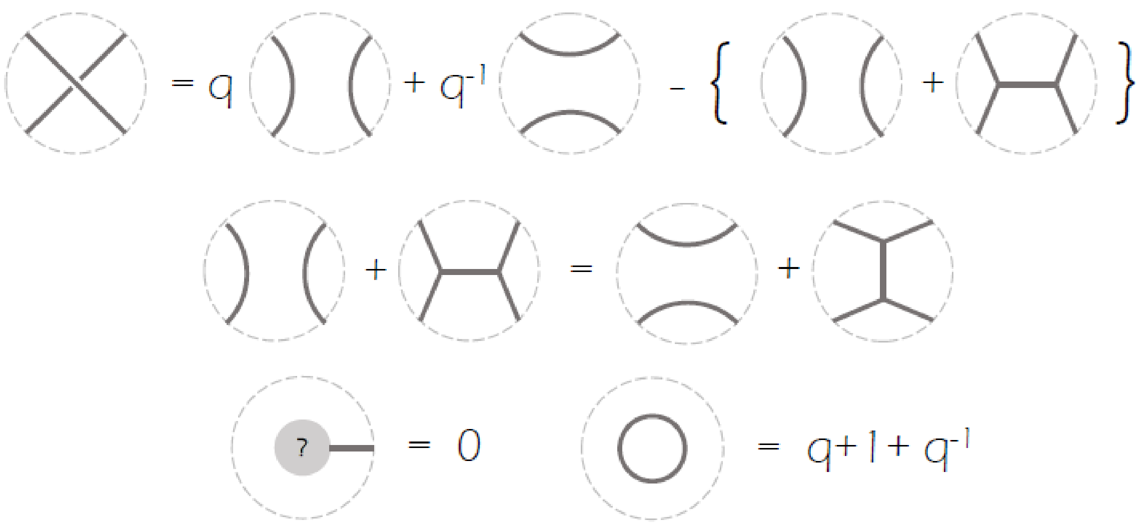
Then how do we compute those link invariants? We'll use cutting and gluing again! Let's take $G = SU(2)$ and $R$ = spin $\frac{1}{2}$ representation as the simplest example. [6] The Hilbert space $\mathcal{H}_{\Sigma_{0,4}}$ of a sphere with 4 marked points is 2-dimensional. (This is roughly due to the fact that $\mathrm{Hom}(2 \otimes 2 \otimes 2 \otimes 2, \mathbb{C})$ is 2-dimensional.) Hence the following three states are linearly dependent

Indeed,

$$q^{1/4} J_{L_+} - q^{-1/4} J_{L_-} = (q^{1/2} - q^{-1/2}) J_{L_0}$$

where $q = e^{\frac{2\pi i}{k+2}}$. This $J_L \in \mathbb{Z}[q^{1/4}, q^{-1/4}]$ is called the Jones polynomial.

Similarly one can define invariants of (colored, framed) spatial webs. For example, in case $G = SU(2)$ and $R$ = spin 1 representation, $\mathrm{Hom}(3 \otimes 3 \otimes 3 \otimes 3, \mathbb{C})$ is 3-dimensional, and hence we have the following 4-term skein relations :




In the classical limit $q \longrightarrow 1$, this invariant just counts the number of Tait colorings; it becomes insensitive to the choice of embedding.

# 3 Aspects of CS theory

## 3.1 Perturbative CS, torsion, Vassiliev invariants

By perturbing the path integral definition of WRT invariant, it was conjectured by Witten [7] that

$$Z_k(M) \sim \frac{1}{2} e^{-3\pi i/4} \sum_{\alpha} \sqrt{T_\alpha(M)} e^{-2\pi i I_\alpha/4} e^{2\pi i (k+2) CS(A)}$$

asymptotically, as $k \longrightarrow \infty$ where $\alpha$ ranges over (connected components of) flat connections, $T_\alpha$ is the Ray-Singer torsion (or Reidemeister torsion) of $M$, and $I_\alpha \in \mathbb{Z}/8\mathbb{Z}$.

In case of links in $S^3$, by perturbing, we can get Vassiliev invariants (= finite type invariants) of links. See [3].

[5] This is because $J(H; \lambda, \mu) = \frac{S_{\lambda\mu}}{S_{00}}$ and $\frac{J(U_1;\lambda)}{J(U_0;\lambda)} = T_{\lambda\lambda}$.

[6] Representations of $SU(2)$ are self-dual, so we don't need to take care of orientation of links in this case.

### 3.2 Complex CS theory, volume conjecture, AJ conjecture

Let $K$ be a knot in $S^3$, and let $M = S^3 \setminus K$ be the complement with torus boundary $\partial M = \Sigma_1$. The classical phase space for $SL(2,\mathbb{C})$ CS theory is the moduli space of flat $SL(2,\mathbb{C})$-connections on $\Sigma_1$, which is

$$\mathcal{M}_{\Sigma_1} \simeq (\mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{Z}/2\mathbb{Z})$$

where $\mathbb{Z}/2\mathbb{Z}$ acts by $(x,y) \mapsto (\frac{1}{x}, \frac{1}{y})$. [9] It has a natural symplectic form

$$\omega = d\ln x \wedge d\ln y$$

The moduli space $\mathcal{M}_M$ of flat $SL(2,\mathbb{C})$-connections on $M$ sits inside $\mathcal{M}_{\Sigma_1}$ as a Lagrangian submanifold, and it is an algebraic curve; it can be described by a polynomial equation $A(x,y) = 0$. Such polynomial is called the *A-polynomial* for the knot $K$. The A-polynomial is a function on the phase space $\mathcal{M}_{\Sigma_1}$. After quantization, it becomes an operator $\hat{A}$ on the Hilbert space $\mathcal{H}_{\Sigma_1}$, and it should annihilates the vector $Z(M) \in \mathcal{H}_{\Sigma_1}$ :

$$\hat{A}\, Z(M) = 0$$

The *AJ conjecture* (a.k.a. quantum volume conjecture) states that both $SU(2)$ and $SL(2,\mathbb{C})$ partition functions must satisfy the quantum constraint equation. This in turn gives us a $q$-difference equation between colored Jones polynomials, hence relating the A-polynomial and the colored Jones polynomials.

### 3.3 3d-3d correspondence, "WRT blocks", categorification

Witten [12] gave a physical description of Khovanov homology, a categorification of the Jones polynomial. Building upon that idea, there are some recent ideas and proposals from physics [13, 14] which could possibly lead to categorification of WRT invariants. Apparently, "*6d $\mathcal{N} = (2,0)$ SCFT of $A_1$ type*" on $M \times D^2 \times \mathbb{R}$ with boundary condition $a$ on $\partial D^2$ plays the central role in this field. By compactifying $M_3$, we get a 3d $\mathcal{N} = 2$ theory $T[M]$ on $D^2 \times \mathbb{R}$, and such theory $T[M]$ (as well as partition functions and indices of the theory) is an invariant of the 3-manifold $M$. Such correspondence is called the *3d-3d correspondence*.

One particular invariant of our interest is the supersymmetric partition function of $T[M]$ on $D^2 \times_q S^1$ :

$$\hat{Z}_a(q) = Z_{T[M]}(D^2 \times_q S^1; a) \in 2^{-c} q^{\Delta_a} \mathbb{Z}[[q]]$$

These "*WRT blocks*" are conjectured to be building blocks for the WRT invariant. For instance, when $M$ is a plumbed 3-manifold (i.e. when $M$ has a tree-shaped surgery link) the WRT invariant can be decomposed into the following form, possibly up to some phase factor :

$$Z_{SU(2)_k}[M] = (i\sqrt{2k})^{b_1(M)-1} \sum_{a,b} e^{2\pi i k\, \ell k(a,a)} S_{ab} \hat{Z}_b(q)\big|_{q \to e^{\frac{2\pi i}{k}}}$$

where $a, b \in \operatorname{Tor} H_1(M,\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})$ and $S_{ab}$ is a sort of a Fourier transform. [7]

Moreover, physics [13, 14] predicts that the $Q$-cohomology (= BPS spectrum) of the Hilbert space of $T[M]$ doubly graded :

$$\mathcal{H}_{T[M]}(D^2; a) = \bigoplus_{\substack{i \in \mathbb{Z}+\Delta_a \\ j \in \mathbb{Z}}} \mathcal{H}_a^{i,j}$$

where the $q$-grading $i$ corresponds to the charge under $U(1)_q$ rotation of $D^2$, and the homological grading $j$ corresponds to the $R$-charge of the $R$-symmetry $U(1)_R$. This BPS spectrum categorifies $\hat{Z}_a(q)$; i.e.

$$\hat{Z}_a(q) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}_a^{i,j}$$

[7] When $M$ is not a plumbed 3-manifold, it is not clear yet what $a$ and $b$ should be indexed by.

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